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A cosmological characteristic initial value problem

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Abstract. An initial value problem for a Robertson–Walker universe filled with perfect fluid is considered by specifying data on an initial null hypersurface. The approach is via techniques developed by Bondi, van der Burg and Metzner as adapted by Chellone and Williams to deal with a perfect fluid space–time. It is shown that it is, in principle, possible to obtain a solution to a more general problem with perturbations away from the spherically symmetric Robertson–Walker universe.

1. Introduction

The characteristic initial value problem in general relativity has been considered by a number of authors, including Bondi *et al* (1962), Sachs (1962), Chellone and Williams (1973, to be referred to as CW) and d'Inverno (1975). In this paper the cosmological characteristic initial value problem for a homogeneous, isotropic, universe filled with perfect fluid will be discussed. The method used here is to begin with the Robertson–Walker metric and then to transform this metric into the form associated with Bondi *et al* (1962), which is expressed in terms of coordinates based on a family of null geodesics. Once this has been demonstrated an analysis of the characteristic initial value problem for a perfect fluid given in CW can be adapted and the solution to the cosmological problem considered.

Although the analysis in CW dealt with the axially and azimuth reflection symmetric case, no difficulty, in principle, was foreseen in extending the work to the general case without any symmetries (Sachs 1962); so one of the advantages of the techniques used here is that it is, in principle, possible to build in perturbations away from spherical symmetry into a description of a Robertson–Walker universe. Moreover as data are specified on a null hypersurface, the problems of perturbations dying away with time, which occur when data are given on a space-like hypersurface (Hawking 1969), are, to some extent, avoided.

The Robertson–Walker solutions are briefly reviewed in § 2 together with a treatment of the initial value problem where data are specified on a space-like hypersurface. Section 3 contains the transformation of the Robertson–Walker metric into the Bondi form and the characteristic initial value problem is considered in § 4. Some conclusions are presented in § 5.

2. The Robertson–Walker solutions

The Robertson–Walker solution describing a homogeneous, isotropic universe is given by the metric (Weinberg 1972)

$$ds^2 = dt^2 - R(t)^2\left(\frac{dr^2}{1-kr^2} + d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{2.1}$$

where the coordinates are (t, r, θ, ϕ), $R(t)$ is the 'scale factor' for the universe and k is $-1, 0$ or $+1$. If the universe is filled with perfect fluid the energy–momentum tensor is

$$T_{\mu\nu} = (p+\rho)v_\mu v_\nu - pg_{\mu\nu}$$

where p, ρ and v^μ are, respectively, the pressure, density and velocity vector for the fluid.

For this situation the field equations and conservation conditions reduce to (Weinberg 1972)

$$\left(\frac{dR}{dt}\right)^2 + k = \frac{8\pi G}{3}\rho R^2 \tag{2.2}$$

and

$$\frac{d}{dR}(\rho R^3) = -3pR^2. \tag{2.3}$$

An integration scheme for the formal solution of the initial value problem can now be given. Suppose ρ is given, as a function of r, on an initial space-like hypersurface and the equation of state in the form $p = p(\rho)$ is known. R can be determined from equation (2.3) which then enables dR/dt to be obtained from equation (2.2) provided k is known. On the 'next' hypersurface the new value of R is used to obtain ρ from the integrated version of (2.3). So the initial data consist of ρ, k, and the equation of state. Although this is a simple and straightforward analysis the coordinate system employed has some inherent disadvantages:

(i) The coordinate system is one in which the contents of the universe are always at rest, so the velocity vector does not appear in the solution of the problem.

(ii) It is not usually convenient, from a practical standpoint, to give data on a space-like hypersurface as all our observational data are propagated along null rays.

Accordingly it seems worthwhile to consider the equivalent characteristic initial value problem, where data are specified on a null hypersurface. The analysis relies heavily on the results and procedures of Bondi *et al* (1962) and CW.

3. Transformation into null coordinates

The Robertson–Walker solution (2.1) will now be transformed into the Bondi form of metric

$$ds^2 = \left(\frac{V e^{2\beta}}{r'} - U^2 r'^2 e^{2\gamma}\right) du'^2 + 2 e^{2\beta} du'\, dr' + 2Ur'^2 e^{2\gamma} du'\, d\theta' - r'^2(e^{2\gamma} d\theta'^2 + e^{-2\gamma} \sin^2\theta'\, d\phi'^2) \tag{3.1}$$

where U, V, β and γ are, in general, functions of the coordinates u', r' and θ'. These coordinates, and the ϕ' coordinate, are based on a family of null hypersurfaces parametrized by u', r' is a luminosity distance along the null geodesics generating the hypersurfaces and θ' and ϕ' label these geodesics. The metric (3.1) is a general axially and azimuth (ϕ) reflection symmetric metric. To make this spherically symmetrical it is apparent that V and β must be functions of u and r only and U and γ must vanish. As the Robertson–Walker metric is isotropic as well as homogeneous there should be a further specialization of the Bondi metric, but this does not appear to take on a simple form. The approach that will be adopted here is to use the spherically symmetric Bondi metric and let the isotropy make itself apparent through the transformation procedure.

Bearing this in mind the transformation of (2.1) into the form (3.1) is taken to be

$$t = u' + f(r') = a(u', r')$$

$$r = r'/R(a(u', r'))$$

$$\theta = \theta'$$

$$\phi = \phi'.$$

The relation giving the new (Bondi) metric coefficients in terms of the old (Robertson–Walker) ones is

$$g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^{\rho}}{\partial x^{\mu'}} \frac{\partial x^{\sigma}}{\partial x^{\nu'}}.$$

The first requirement for the transformed metric is $g'_{11} = 0$; this is

$$f'^2 - \frac{R(a(u', r'))}{1 - kr'^2/R^2} \left(R^{-1} - r'R^{-2} \frac{\mathrm{d}R}{\mathrm{d}a} f' \right)^2 = 0, \tag{3.2}$$

where $f' = \mathrm{d}f/\mathrm{d}r$, which enables f' to be determined in terms of $R(a)$ and k. The other requirements are that $g'_{12} = g'_{13} = 0$, which are identically satisfied.

The remaining non-zero metric coefficients are

$$g'_{00} = 1 - \frac{R}{1 - kr'^2/R^2} \left(-r'R^{-2} \frac{\mathrm{d}R}{\mathrm{d}a} \right)^2 \tag{3.3}$$

$$g'_{01} = f' - \frac{R}{1 - kr'^2/R^2} \left(-r'R^{-2} \frac{\mathrm{d}R}{\mathrm{d}a} \right) \left(R^{-1} - r'R^{-2} \frac{\mathrm{d}R}{\mathrm{d}a} f' \right) \tag{3.4}$$

$$g'_{22} = -r'^2 \tag{3.5}$$

$$g'_{33} = -r'^2 \sin^2\theta. \tag{3.6}$$

Identifying g'_{01} with $e^{2\beta}$ and g'_{00} with $V e^{2\beta}/r'$, equations (3.3) and (3.4) reduce to a relation between V and β, provided, as before, $R(a)$ and k are known. This relation corresponds to the isotropy condition previously referred to. However, as far as the initial value problem is concerned this relation presupposes a knowledge of $R(a)$ and $\mathrm{d}R/\mathrm{d}a$ which would prejudice the solution to the problem. The approach followed here will be to treat β and V as independent variables, with the understanding that when their values are obtained on any particular null hypersurface, the values of R and $\mathrm{d}R/\mathrm{d}a$ on that hypersurface may be obtained, if required, by solving equations (3.3) and (3.4).

One further equation that will be required for a consideration of the initial value problem is that due to the normalization of the velocity vector. This equation

$$v_\mu v^\mu = 1$$

reduces, for the case of spherical symmetry, to

$$v_0 = \tfrac{1}{2} v_1 V r^{-1} + \tfrac{1}{2} e^{2\beta} v_1^{-1} \tag{3.7}$$

where the primes on the null coordinates are omitted, as they will be in the succeeding sections.

4. The characteristic initial value problem

To take account of the field equations $G_{\mu\nu} = \frac{8}{3}\pi G T_{\mu\nu}$, and the conservation condition $T^{\mu\nu}{}_{;\nu} = 0$ the techniques developed by Bondi *et al* (1962) as adapted by Chellone and Williams (1973) will be applied (see also d'Inverno 1975). These techniques result in the field equations being split into two categories, those with and without terms containing u derivatives.

If $E_{\mu\nu} = G_{\mu\nu} - \frac{8}{3}\pi G T_{\mu\nu}$, then the equations without u derivatives corresponding to $E_{11} = 0$ and $g^{11}E_{11} + 2g^{01}E_{01} + 2g^{12}E_{12} = 0$, respectively, are

$$\frac{4\beta_{,1}}{r} - \frac{8\pi G}{3}(p+\rho)v_1^2 = 0 \tag{4.1}$$

and

$$\frac{-2\,e^{-2\beta}V_{,1}}{r^2} + \frac{2}{r^2} + \frac{8\pi G}{3}(p-\rho) = 0. \tag{4.2}$$

The equation corresponding to $E_{12} = 0$ is now identically satisfied. If, now, the initial data are chosen to be ρ, v_1 and an equation of state, and these data are suitably continuous and specified on a null hypersurface u = constant, equation (4.1) determines β on the hypersurface, up to a constant of integration. Equation (4.2) can then be used to obtain the value of V, again up to a constant of integration.

The equations with u derivatives, corresponding to $T_0{}^{\nu}{}_{;\nu} = 0$ and $T_1{}^{\nu}{}_{;\nu}$ are, respectively

$$\rho_{,0}[(1+p')v_0 v_1\, e^{-2\beta} - p'] + v_{1,0}\, e^{-2\beta}(p+\rho)v_1 V r^{-1} = J_0 \tag{4.3}$$

and

$$(1+p')\rho_{,0} v_1^2\, e^{-2\beta} + v_{1,0} 2(p+\rho)v_1\, e^{-2\beta} = J_1 \tag{4.4}$$

where $p' = \mathrm{d}p/\mathrm{d}\rho$ and J_0 and J_1 stand for functions already known on the hypersurface. The equations corresponding to $g^{22}E_{22} - g^{33}E_{33} = 0$ and $T_2{}^{\nu}{}_{;\nu} = 0$ are identically satisfied.

Equations (4.3) and (4.4) can be solved for $\rho_{,0}$ and $v_{1,0}$ if

$$(p+\rho)(1-p')v_1\, e^{-2\beta} \neq 0.$$

Accordingly the following two assumptions are made: (i) $p+\rho \neq 0$; (ii) $p' \neq 1$. This last assumption corresponds to requiring that the speed of sound in the fluid is not equal to the speed of light. It is a direct consequence of the normalization condition (3.7) that $v_1 \neq 0$.

With these assumptions the values of the data variables can be calculated on the 'next' hypersurface, and then, in turn, the values of the metric variables obtained on that hypersurface.

The remaining non-trivial field equation corresponds to $E_{00}=0$, and is

$$\frac{2\beta_{,1}V^2}{r^3}+\frac{V_{,0}}{r^2}+\frac{V\,e^{2\beta}}{r^3}-\frac{VV_{,1}}{r^3}-\frac{2V\beta_{,0}}{r^2}-\frac{8\pi G}{3}(p+\rho)v_0^2-\frac{8\pi G}{3}p\frac{V}{r}\,e^{2\beta}=0. \qquad (4.5)$$

It can be shown (CW) that (4.5) is equivalent to requiring that

$$E_{00}=h(u,\theta)/r^2$$

where h is arbitrary, and it turns out that this can only be satisfied, in general, if the metric and physical variables are expanded in positive powers of r.

As a result of this expansion technique and the imposition of continuity conditions on $r=0$, it was shown that the constant of integration occurring in the equation for β (4.1) is non-zero and that occurring in (4.2) is equal to zero.

It has now been shown that on an initial null hypersurface, data can be specified that entirely determine the behaviour of the Robertson–Walker universe at future times. These data are essentially ρ, v_1 and an equation of state $p=p(\rho)$ together with one arbitrary function of integration $\overset{0}{\beta}(u)$. In detail owing to the expansion technique used, only certain coefficients of these variables are independent, and referring to CW, if

$$\rho=\overset{0}{\rho}+rF+O(r^2)$$

$$p=\overset{0}{p}+rD+O(r^2)$$

$$v_1=rb+O(r^2)$$

where all coefficients are functions of u alone, only, say, $\overset{0}{\rho}$ and F need be specified while $b=0$.

5. Conclusion

A formal solution to the characteristic initial value problem for a perfect-fluid-filled Robertson–Walker universe has been shown to exist, provided the metric and physical variables are expanded in positive powers of a radial parameter. A, perhaps, unsatisfactory aspect of this solution is the loss of the specific relation corresponding to the property of isotropy of the Robertson–Walker universe, although this can be recovered when the functional dependence of the 'scale factor' $R(t)$ is known.

To extend this analysis to the axially and azimuth reflection symmetric case all that is required is to give, in addition, the angular dependence of the above data variables together with the appropriate coefficients of Bondi's γ and v_2 and another function of u integration. In this way perturbations away from spherical symmetry can be built into the Robertson–Walker solutions, and an extension to make these perturbations completely general, without any symmetries, is perfectly feasible, although computationally more complex.

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